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Magnetic polarisability of small apertures: analytical approach

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Abstract. A new method is proposed for the analysis of the magnetic polarisability of small apertures of arbitrary shape. The method is based on an integral represenation for the reciprocal distance between two points previously obtained by the author. A general formula is derived for the coefficients of magnetic polarisability of small apertures. Specific formulae are obtained for the apertures shaped as a polygon, a triangle, a rectangle, a rhombus, a circular sector and a circular segment. All the formulae are checked against the solutions known in the literature and their accuracy is confirmed.

1. Introduction

Many years ago Bethe (1944) reduced the problem of diffraction by small apertures to an evaluation of the coefficient of electric polarisability and the tensor of magnetic polarisability. At the moment, closed-form expressions for these quantities are known for an elliptic aperture in a planar screen only. All non-elliptic shapes have been treated either experimentally (Cohn 1951) or numerically (Okon and Harrington 1981, de Smedt 1979, De Meulenaere and Van Bladel 1977); the variational approach was used by Fikhmanas and Fridberg (1973). Though their results sometimes differ by more than the accuracy they claim, we have no other source for verification of the accuracy of the formulae to be derived here.

The theory related to the new analytical approach is discussed in the next section. Some general approximate formulae are derived for the components of the tensor of magnetic polarisability which are valid for an aperture of arbitrary shape. The accuracy of the general formulae cannot be verified at the moment since there are neither experimental nor numerical data available. A significant simplification occurs when the aperture has at least one axis of symmetry: the tensor of magnetic polarisability becomes diagonal. Specific formulae for the evaluation of the coefficients of magnetic polarisability are derived for various aperture shapes and their accuracy proves to be quite satisfactory when compared with the numerical results available. The second part of the project will deal with the coefficients of electric polarisability.

2. Theory

It is well known (Bethe 1944) that the problem of diffraction by small apertures can be reduced to the solution of the following integral equation:

$$w(N) = \int\int_S \frac{\sigma(M)}{R(M,N)} \, dS \tag{1}$$

where S is a two-dimensional domain of the aperture, $R(M, N)$ stands for the distance between the points M and N, w is a known function and σ stands for the charge density (unknown function).

Here we outline the analytical treatment of the problem which allows us to derive simple yet accurate formulae for various aperture shapes. The approach is based on the integral representation for the reciprocal distance established by Fabrikant (1971)

$$\frac{1}{[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)]^{1/2}}=\frac{2}{\pi}\int_0^{\min(\rho_0,\rho)}\frac{\lambda(x^2/\rho\rho_0, \phi-\phi_0)\,\mathrm{d}x}{[(\rho^2-x^2)(\rho_0^2-x^2)]^{1/2}} \tag{2}$$

where

$$\lambda(k, \psi)=\frac{1-k^2}{1+k^2-2k\cos\psi}. \tag{3}$$

Since the derivation of (2) was published in Russian and is not easily accessible to the reader, we repeat this derivation in the appendix. Substitution of (2) into (1) gives, after changing the order of integration,

$$w(\rho, \phi)=\frac{2}{\pi}\int_0^{\rho}\frac{\mathrm{d}x}{(\rho^2-x^2)^{1/2}}\int_0^{2\pi}\mathrm{d}\phi_0\int_x^{a(\phi_0)}\frac{\lambda(x^2/\rho\rho_0, \phi-\phi_0)}{(\rho_0^2-x^2)^{1/2}}\sigma(\rho_0, \phi_0)\rho_0\,\mathrm{d}\rho_0. \tag{4}$$

Consider an aperture S in a planar screen whose boundary is given in the polar coordinates as

$$\rho=a(\phi)$$

where the function $a(\phi)$ is bounded and single-valued. For the case of magnetic polarisability, it is sufficient to consider equation (1), with the function w taking the form

$$w=\alpha_x y-\alpha_y x \tag{5}$$

where α_x and α_y are constant. It is quite clear that in the case of a uniaxial excitation one of these constants can be put equal to zero.

Let the charge distribution at the aperture be

$$\sigma(\rho, \phi)=\frac{a(\phi)\rho(p_1\cos\phi+p_2\sin\phi)}{[a^2(\phi)-\rho^2]^{1/2}} \tag{6}$$

where p_1 and p_2 are as yet unknown constants. The main reason for this choice is the requirement that (6) be exact for an ellipse. We make use of the condition that the integral of σ over S should be equal to zero. Since p_1 and p_2 are independent, this leads to two equations

$$\int_0^{2\pi}(a(\phi))^3\cos\phi\,\mathrm{d}\phi=0 \qquad \int_0^{2\pi}(a(\phi))^3\sin\phi\,\mathrm{d}\phi=0. \tag{7}$$

One can note that the left-hand side of each equation (7) is proportional to the x or y coordinates of the centre of gravity. This means that the origin of the system of coordinates should be located at the centre of gravity of the aperture. The axis orientation will be discussed later.

The relationsips between the dipole moments and the parameters p_1 and p_2 can be established from the conditions

$$M_x=\iint_S\sigma y\,\mathrm{d}S \qquad M_y=-\int_S\sigma x\,\mathrm{d}S$$

which leads to

$$M_x = \tfrac{8}{3}(p_1 I_{xy} + p_2 I_x) \qquad M_y = -\tfrac{8}{3}(p_1 I_y + p_2 I_{xy}) \tag{8}$$

where I_x, I_y and I_{xy} are the well known quantities of the moments of inertia and the product of inertia respectively:

$$I_x = \iint_S y^2\, dS \qquad I_y = \iint_S x^2\, dS \qquad I_{xy} = \iint_S xy\, dS.$$

Now it is necessary to relate p_1 and p_2 to the parameters α_x and α_y. This can be done by substitution of (6) into (4) which yields, after integration with respect to ρ_0,

$$w(\rho, \phi) = \sum_{n=-\infty}^{\infty} \int_0^{\rho} \left(\frac{x}{\rho}\right)^{|n|} \frac{x^2\, dx}{(\rho^2 - x^2)^{1/2}} \int_0^{2\pi} \exp[in(\phi - \phi_0)]$$
$$\times F\left(\frac{3-|n|}{2}, \frac{1}{2}; 1; 1 - \frac{x^2}{a^2(\phi_0)}\right) (p_1 \cos \phi_0 + p_2 \sin \phi_0)\, d\phi_0. \tag{9}$$

Here F stands for the Gauss hypergeometric function. Further evaluation of the function w can be done separately for each harmonic. Note that the zeroth and all the even harmonics of w will be zero if $a(\phi)$ contains only the even harmonics. The first harmonic will take the form

$$w_1(\rho, \phi) = \frac{\pi}{2} \rho \int_0^{2\pi} \cos(\phi - \phi_0)(p_1 \cos \phi_0 + p_2 \sin \phi_0) a(\phi_0)\, d\phi_0$$

which can be simplified as

$$w_1(\rho, \phi) = \tfrac{1}{2}\pi\rho[(p_1 J_y + p_2 J_x y) \cos \phi + (p_1 J_{xy} + p_2 J_x) \sin \phi] \tag{10}$$

where the following quantities were introduced:

$$J_x = \int_0^{2\pi} a(\phi) \sin^2 \phi\, d\phi \qquad J_y = \int_0^{2\pi} a(\phi) \cos^2 \phi\, d\phi$$
$$J_{xy} = \int_0^{2\pi} a(\phi) \sin \phi \cos \phi\, d\phi. \tag{11}$$

These quantities do not seem to have been used before in engineering practice so they do not have an accepted name. Since their tensor properties are similar to those of the moments of inertia, we shall call J_x and J_y *the linear moments of a two-dimensional domain about the axes Ox and Oy* respectively, J_{xy} will be called *the linear product of a two-dimensional domain about the axes Ox and Oy.*

It is important to note that the third harmonic is equal to zero for an arbitrary contour. Here is the expression for the fifth harmonic

$$w_5(\rho, \phi) = \frac{128}{315} \rho^4 \int_0^{2\pi} \frac{\cos 5(\phi - \phi_0)}{a^2(\phi_0)} (p_1 \cos \phi_0 + p_2 \sin \phi_0)\, d\phi_0$$

which can be modified as

$$w_5(\rho, \phi) = \tfrac{64}{315}\rho^4\{[(A_{c6} + A_{c4}) p_1 + (A_{s6} - A_{s4}) p_2] \cos 5\phi$$
$$+ [(A_{s6} + A_{s4}) p_1 + (A_{c4} - A_{c6}) p_2] \sin 5\phi\}. \tag{12}$$

Here, the following geometrical characteristics of the aperture domain were introduced

$$A_{c4}=\int_0^{2\pi}\frac{\cos 4\phi\ \mathrm{d}\phi}{(a(\phi))^2}\qquad A_{c6}=\int_0^{2\pi}\frac{\cos 6\phi\ \mathrm{d}\phi}{(a(\phi))^2}$$

$$A_{s4}=\int_0^{2\pi}\frac{\sin 4\phi\ \mathrm{d}\phi}{(a(\phi))^2}\qquad A_{s6}=\int_0^{2\pi}\frac{\sin 6\phi\ \mathrm{d}\phi}{(a(\phi))^2}.$$

Investigation of further harmonics shows that their amplitude decreases.

Now consider in more detail the case of a square with side $2l$. The equation of the boundary in this case is $a(\phi)=l/\cos\phi$ for $-\pi/4<\phi<\pi/4$, and the pattern is repeated outside this range. We can evaluate the first two non-zero harmonics:

$$w_1=\pi l\rho\ \ln(1+\sqrt{2})(p_1\cos\phi+p_2\sin\phi)$$
$$w_5=\frac{128\rho^4}{945l^2}(p_1\cos 5\phi+p_2\sin 5\phi). \qquad (13)$$

Since the amplitude of w_5 is significantly less than that of w_1, it seems natural to assume $w\approx w_1$, and the remaining harmonics may be called the solution error. Direct computations show that the error is less than 3% inside the circle $\rho\leqslant l$. The error is reasonably small outside the circle, reaching about 20% at the apex and decreasing very rapidly with distance from the apex. Taking into consideration that the error sign fluctuation will result in an even smaller error in the integral characteristics sought, a direct comparison of (5) and (10) leads to

$$\alpha_x=\tfrac{1}{2}\pi(p_1J_{xy}+p_2J_x)\qquad \alpha_y=-\tfrac{1}{2}\pi(p_1J_y+p_2J_{xy}). \qquad (14)$$

The inversion of (14) gives

$$p_1=-\frac{2}{\pi}\frac{J_{xy}\alpha_x+J_x\alpha_y}{J_xJ_y-J_{xy}^2}\qquad p_2=\frac{2}{\pi}\frac{J_y\alpha_x+J_{xy}\alpha_y}{J_xJ_y-J_{xy}^2}. \qquad (15)$$

Substitution of (15) in (8) finally gives the required relationship

$$M_x=\frac{16}{3\pi}(m_{11}\alpha_x+m_{12}\alpha_y)\qquad M_y=\frac{16}{3\pi}(m_{21}\alpha_x+m_{22}\alpha_y) \qquad (16)$$

where

$$m_{11}=\frac{J_yI_x-J_{xy}I_{xy}}{J_xJ_y-J_{xy}^2}\qquad m_{12}=\frac{J_{xy}I_x-J_xI_{xy}}{J_xJ_y-J_{xy}^2}$$

$$m_{21}=\frac{J_{xy}I_y-J_yI_{xy}}{J_xJ_y-J_{xy}^2}\qquad m_{22}=\frac{J_xI_y-J_{xy}I_{xy}}{J_xJ_y-J_{xy}^2}.$$

It is clear that all these results can be rewritten in a matrix or a tensor form. One can verify that formulae (16) are invariant with respect to an arbitrary rotation of the axes. The same property holds for $m_{11}+m_{22}$ and $m_{12}-m_{21}$. Strictly speaking, according to the reciprocal theorem, m_{12} should equal m_{21}, so that formulae (16) generally do not satisfy this theorem, but we may state that this theorem is satisfied 'approximately'. We mean by this the following property which has been verified by several direct computations, namely $|m_{12}-m_{21}|/m_{11}\ll 1$ and $|m_{12}-m_{21}|/m_{22}\ll 1$. This theorem will be satisfied exactly for any domain which has at least one axis of symmetry because in this case $m_{12}=m_{21}=0$ provided that the coordinate axes coincide with the central principal axes of the domain of contact. Since we have no numerial data for non-symmetrical domains which could be used to verify the accuracy of (16), we shall consider further only the case when the aperture has an axis of symmetry. In this case formulae (8), (14) and (16) simplify significantly

$$M_x = \tfrac{8}{3} I_x p_2 \qquad M_y = -\tfrac{8}{3} I_y p_1 \tag{17}$$

$$\alpha_x = \tfrac{1}{2}\pi J_x p_2 \qquad \alpha_y = -\tfrac{1}{2}\pi J_y p_1 \tag{18}$$

$$M_x = \frac{16}{3\pi}\frac{I_x}{J_x}\alpha_x \qquad M_y = \frac{16}{3\pi}\frac{I_y}{J_y}\alpha_y . \tag{19}$$

Now, we can rewrite the expression for the charge distribution (6) in terms of the moments M_x and M_y in the form

$$\sigma = \frac{a(\phi)}{2[a^2(\phi) - \rho^2]^{1/2}}\left[\frac{3}{4}\left(\frac{M_x y}{I_x} - \frac{M_y x}{I_y}\right)\right]. \tag{20}$$

An expression equivalent to (20) can be written in terms of the parameters α_x and α_y

$$\sigma = \frac{2a(\phi)}{\pi[a^2(\phi) - \rho^2]^{1/2}}\left(\frac{\alpha_x y}{J_x} - \frac{\alpha_y x}{J_y}\right). \tag{21}$$

Expressions (20) and (21) are *exact* for an ellipse. We expect them to be reasonably accurate in the neighbourhood of the coordinate origin for an arbitrary aperture with at least one axis of symmetry, while the error might become quite significant close to the boundary of the domain S.

Let us rewrite formulae (19) in the form

$$M_x = \frac{A^{3/2}}{2\pi}\nu_x \alpha_x \qquad M_y = \frac{A^{3/2}}{2\pi}\nu_y \alpha_y \tag{22}$$

when A is the aperture area, and

$$\nu_x = \frac{32 I_x}{3A^{3/2} J_x} \qquad \nu_y = \frac{32 I_y}{3A^{3/2} J_y} . \tag{23}$$

We introduced the coefficients ν_x and ν_y for two reasons: since they are dimensionless they characterise the shape of S and do not depend on its size; both ν_x and ν_y are equal to the corresponding coefficients of magnetic polarisability which will simplify the comparison of our results with the numerical data available. The remaining part of this paper will be devoted to the evaluation of the coefficients ν_x and ν_y for various aperture shapes.

3. Applications

Several specific aperture shapes are considered here. Each configuration is related to its central principal axes and assumed to have at least one axis of symmetry coinciding with the axis Ox. A high degree of accuracy of formulae (23) is confirmed by comparison with available numerical solutions.

3.1. Polygon

Consider a polygon with n sides. The function $a(\phi)$ describing its boundary is bounded and single-valued. The origin of the coordinate system is located at the centre of gravity, as before. Let us number the polygon sides in a counterclockwise direction from 1 to n, a_k being the length of the kth side. The apex, at which the sides a_k and a_{k+1} are intersecting, is numbered $k+1$. It is clear that the value of index equal $n+1$ is understood as 1. Denote by b_k the distance from the centre of gravity to the kth apex; ψ_k stands for the angle between the axis Ox and the perpendicular to the side a_k. Let A_k be the area of the triangle formed by a_k, b_k and b_{k+1}, the total area A of the polygon being equal to the sum of A_k. The following expressions can be obtained for the moments of inertia:

$$I_x = \sum_{k=1}^{n} -m_k \cos 2\psi_k + g_k \sin 2\psi_k + 2h_k \cos^2 \psi_k$$

$$I_y = \sum_{k=1}^{n} m_k \cos 2\psi_k - g_k \sin 2\psi_k + 2h_k \sin^2 \psi_k \tag{24}$$

$$I_{xy} = \sum_{k=1}^{n} (m_k - h_k) \sin 2\psi_k + g_k \cos 2\psi_k$$

where

$$m_k = \frac{2A_k^3}{a_k^2} \qquad g_k = A_k^2 \frac{b_{k+1}^2 - b_k^2}{2a_k^2} \qquad h_k = \frac{A_k[3(b_{k+1}^2 + b_k^2) - a_k^2]}{24}. \tag{25}$$

Formulae (24) and (25) are valid for an arbitrary polygon, not necessarily having an axis of symmetry. The principal moments of inertia I_{xc} and I_{yc} and the principal axes orientation angle ψ_c can be computed due to the well known formulae (see D'Souza and Garg 1984)

$$I_{x,yc} = \frac{I_x + I_y}{2} \pm \left[\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2\right]^{1/2} \qquad \psi_c = \tfrac{1}{2}\tan^{-1}\frac{2I_{xy}}{I_x - I_y}$$

where

$$\frac{I_x + I_y}{2} = \sum_{k=1}^{n} h_k \qquad \frac{I_x - I_y}{2} = \sum_{k=1}^{n} -(m_k - h_k)\cos 2\psi_k + g_k \sin 2\psi_k.$$

The linear moments can be computed in the form

$$J_x = \sum_{k=1}^{n} -q_k \cos 2\psi_k + s_k \sin 2\psi_k + 2t_k \cos^2 \psi_k$$

$$J_y = \sum_{k=1}^{n} q_k \cos 2\psi_k - s_k \sin 2\psi_k + 2t_k \sin^2 \psi_k \tag{26}$$

$$J_{xy} = \sum_{k=1}^{n} (q_k - t_k) \sin 2\psi_k + s_k \cos 2\psi_k$$

where

$$q_k = \frac{A_k}{a_k^2}\left(\frac{1}{b_k} + \frac{1}{b_{k+1}}\right)[a_k^2 + (b_k - b_{k+1})^2] \qquad s_k = 4\frac{A_k^2}{a_k^2}\left(\frac{1}{b_k} - \frac{1}{b_{k+1}}\right)$$

$$t_k = \frac{A_k}{a_k} \ln \frac{b_k + b_{k+1} + a_k}{b_k + b_{k+1} - a_k}. \tag{27}$$

Substitution of (24)-(27) into (23) gives the coefficients ν_x and ν_y for an arbitrary polygon. In the case of a regular polygon $a_k = a$, $b_k = b = a/[2\sin(\pi/n)]$, $\psi_k = 2\pi(k-1)/n$, $A_k = [a^2 \cot(\pi/n)]/4 = [b^2 \sin(2\pi/n)]/2$, $A = nA_k$ and formulae (24)-(27) simplify to

$$I_x = I_y = \frac{na^4}{64} \cot\frac{\pi}{n}\left(\cot^2\frac{\pi}{n} + \frac{1}{3}\right) = \frac{nb^4}{24} \sin\frac{2\pi}{n}\left(2 + \cos\frac{2\pi}{n}\right) \tag{28}$$

$$J_x = J_y = \tfrac{1}{4}na \cot\frac{\pi}{n} \ln\frac{1+\sin(\pi/n)}{1-\sin(\pi/n)} = \tfrac{1}{2}nb \cos\frac{\pi}{n} \ln\frac{1+\sin(\pi/n)}{1-\sin(\pi/n)}. \tag{29}$$

Substituting (28) and (29) in (23) leads to

$$\nu_x = \nu_y = 16[2+\cos(2\pi/n)]\left(9[n^3 \sin(\pi/n)\cos^3(\pi/n)]^{1/2} \ln\frac{1+\sin(\pi/n)}{1-\sin(\pi/n)}\right)^{-1}. \tag{30}$$

Consider several particular values of n. For an equilateral triangle ($n=3$) formula (30) gives $\nu_x = \nu_y = 3^{1/4}16/[27\ln(2+\sqrt{3})] = 0.5922$. We did not find any numerical data to compare with this result. In the case of a square $n = 4$, and $\nu_x = \nu_y = 4/[9\ln(1+\sqrt{2})] = 0.5043$ which is inside the interval from 0.4973 to 0.5162 given by Okon and Harrington (1981) and within 3% from the result of de Smedt, 0.5193. Since formula (30) in the limiting case $n \to \infty$ gives the exact result for a circle $\nu_x = \nu_y = 8/(3\pi^{3/2}) = 0.4789$, we should expect that the error of (30) will decrease with n. The value of the coefficients for a regular hexagon is $\nu_x = \nu_y = 40\sqrt{2}/(3^{1/4}81\ln 3) = 0.4830$ which differs by 1.4% from the result 0.49 due to Okon and Harrington (1981), and it is quite clear that the maximum possible error indeed decreases with n. It is noteworthy that the coefficients of magnetic polarisability do not change significantly in the whole range $3 \leq n < \infty$.

3.2. Isosceles triangle

In the case of a triangle with the sides $a_1 = a_2 = l$ and the angle between them equal to α formulae (23)-(27) give

$$I_x = \tfrac{1}{12}l^4 \sin\alpha \sin^2(\alpha/2) \qquad I_y = \tfrac{1}{36}l^4 \sin\alpha \cos^2(\alpha/2)$$

$$J_x = \frac{2}{3} l \cos\frac{\alpha}{2}\left[\sin\alpha + \sin(\alpha+\gamma) - 2\sin\gamma + 2\sin^3\frac{\alpha}{2} \ln\left(\cot\frac{2\gamma-\alpha}{4}\cot\frac{\alpha}{4}\right) + \ln\tan\left(\frac{\pi}{4}+\frac{\gamma}{2}\right)\right]$$

$$J_y = \frac{2}{3} l \cos\frac{\alpha}{2}\left[-\sin\alpha - \sin(\alpha+\gamma) + 2\sin\gamma + \sin\alpha\cos\frac{\alpha}{2} \ln\left(\cot\frac{2\gamma-\alpha}{4}\cot\frac{\alpha}{4}\right)\right]$$

with the result for the coefficients

$$\nu_x = 8(\tan(\alpha/2))^{3/2}\left\{3\left[\sin\alpha + \sin(\alpha+\gamma) - 2\sin\gamma + 2\sin^3\frac{\alpha}{2} \ln\left(\cot\frac{2\gamma-\alpha}{4}\cot\frac{\alpha}{4}\right) + \ln\tan\left(\frac{\pi}{4}+\frac{\gamma}{2}\right)\right]\right\}^{-1} \tag{31}$$

$$\nu_y = 8(\cot(\alpha/2))^{1/2}\left\{9\left[-\sin\alpha - \sin(\alpha+\gamma) + 2\sin\gamma + \sin\alpha\cos\frac{\alpha}{2}\ln\left(\cot\frac{2\gamma-\alpha}{4}\cot\frac{\alpha}{4}\right)\right]\right\}^{-1}$$

where $\gamma = \tan^{-1}(3\tan(\alpha/2))$.

The isosceles right triangle was considered by Okon and Harrington (1981) who gave the interval between 0.9829 and 1.021 for only one coefficient which in our notation is ν_x. Our result for ν_x is 0.9255 which differs by less than 10% from theirs. The second formula (31) gives $\nu_y = 0.3995$ and there is nothing in the literature to compare with this result.

3.3. *Rectangle*

Consider a rectangular aperture, a_1 and a_2 being its semiaxes. Introduce the aspect ratio $\varepsilon = a_2/a_1$. Formulae (24)-(27) in this case reduce to

$$I_x = \tfrac{4}{3}a_1 a_2^3 \qquad I_y = \tfrac{4}{3}a_1^3 a_2$$

$$J_x = 4a_1 \sinh^{-1}\varepsilon \qquad J_y = 4a_2\sinh^{-1}(1/\varepsilon)$$

and formulae (23) yield

$$\nu_x = \frac{4\varepsilon^{3/2}}{9\sinh^{-1}\varepsilon} \qquad \nu_y = \frac{4\varepsilon^{-3/2}}{9\sinh^{-1}(1/\varepsilon)}. \tag{32}$$

We have found in the literature some numerical results which seem to be more or less accurate. The coefficients of magnetic polarisability were computed by de Smedt (1979) for a rectangle with different aspect ratio ε. We present his results along with those given by (32) in table 1.

Table 1.

ε	0.1000	0.2000	0.3333	0.5000	0.7500	0.8000	1.0000
de Smedt ν_x	0.1287	0.1881	0.2531	0.3249	0.4240	0.4436	0.5193
Formula (32) ν_x	0.1408	0.2001	0.2612	0.3265	0.4165	0.4341	0.5043
de Smedt ν_y	4.1070	2.0260	1.2600	0.8892	0.6426	0.6130	0.5193
Formula (32) ν_y	4.6876	2.1488	1.2701	0.8708	0.6228	0.5929	0.5043
Discrepancy in ν_x (%)	−9.4	−6.4	−3.2	−0.5	1.8	2.2	2.9
Discrepancy in ν_y (%)	−14.1	−6.1	−0.8	2.1	3.1	3.3	2.9

Our formula (32) seems to perform satisfactorily in a sufficiently wide range of aspect ratio. The approximate expression for the charge distribution at the aperture, according to (20), takes the form

$$\sigma = \frac{a(\phi)}{8a_1a_2[a^2(\phi)-\rho^2]^{1/2}}\left[\frac{9}{4}\left(\frac{M_x y}{a_2^2} - \frac{M_y x}{a_1^2}\right)\right]. \tag{33}$$

The results due to (33) can be compared with the numerical data received in a personal communication from de Smedt. In order to make the comparison possible, one should put in (33) $M_x = 0$, replace M_y by (22), with the result

$$\sigma = \frac{9\sqrt{\varepsilon}\,a(\phi)\nu_y x}{4a_1[a^2(\phi)-\rho^2]^{1/2}}. \tag{34}$$

Computations due to (34) were made for $\varepsilon = 0.5$ along the axis Ox, the value ν_y was taken as 0.8708 (see table 1). The results compared to those communicated by de Smedt are shown in table 2.

One should expect the error of (34) to be monotonic (or to have one extremum). This expectation is not met around $x/a_1 = 0.5$ and $x/a_1 = 0.9$ which most probably indicates some computational inaccuracies in the data by de Smedt. This is why we are using the word *discrepancy* rather than the word *error* in the tables throughout the paper. The situation becomes even more evident if we compare the same values along the axis Oy. One can use a formula similar to (34) replacing all x and y and interchanging a_1 and a_2, the value of ν_x was taken to be 0.3265. Changing sign in the discrepancy indicates some 'noise' in the numerical solution by de Smedt.

3.4. *Rhombus*

Let α be the angle at one of the rhombus apexes and l be its side. Formulae (24)-(27) in this case yield

$$I_x = \tfrac{1}{6}l^4 \sin\alpha \sin^2 \tfrac{1}{2}\alpha \qquad I_y = \tfrac{1}{6}l^4 \sin\alpha \cos^2 \tfrac{1}{2}\alpha \qquad A = l^2 \sin\alpha$$

$$J_x = 2l \sin\alpha \left(\cos\tfrac{1}{2}\alpha - \sin\tfrac{1}{2}\alpha + \sin^2\tfrac{1}{2}\alpha \ln \frac{\cos(\alpha/2) + \sin(\alpha/2) + 1}{\cos(\alpha/2) + \sin(\alpha/2) - 1} \right)$$

$$J_y = 2l \sin\alpha \left(-\cos\tfrac{1}{2}\alpha + \sin\tfrac{1}{2}\alpha + \cos^2\tfrac{1}{2}\alpha \ln \frac{\cos(\alpha/2) + \sin(\alpha/2) + 1}{\cos(\alpha/2) + \sin(\alpha/2) - 1} \right).$$

The coefficients will be defined as

$$\nu_x = 8\sin^2(\alpha/2) \left[9(\sin\alpha)^{3/2} \left(\cos\tfrac{1}{2}\alpha - \sin\tfrac{1}{2}\alpha + \sin^2\tfrac{1}{2}\alpha \ln \frac{\cos(\alpha/2) + \sin(\alpha/2) + 1}{\cos(\alpha/2) + \sin(\alpha/2) - 1} \right) \right]^{-1} \tag{35}$$

$$\nu_y = 8\cos^2(\alpha/2) \left[9(\sin\alpha)^{3/2} \left(-\cos\tfrac{1}{2}\alpha + \sin\tfrac{1}{2}\alpha + \cos^2\tfrac{1}{2}\alpha \ln \frac{\cos(\alpha/2) + \sin(\alpha/2) + 1}{\cos(\alpha/2) + \sin(\alpha/2) - 1} \right) \right]^{-1}.$$

Table 2.

x/a_1	0.0833	0.1667	0.2500	0.3333	0.4167	0.5000	0.5833	0.6667	0.7500	0.8333	0.9167
de Smedt σ	0.1143	0.2303	0.3501	0.4759	0.6093	0.7523	0.9367	1.1460	1.4304	1.8303	2.8182
Formula (34) σ	0.1159	0.2342	0.3577	0.4898	0.6350	0.7999	0.9950	1.2392	1.5709	2.0886	3.1777
Discrepancy (%)	−1.3	−1.7	−2.2	−2.9	−4.2	−6.3	−6.2	−8.1	−9.8	−14.1	−12.8

Table 3.

y/a_2	0.1667	0.3333	0.5000	0.6667	0.8333
de Smedt σ	0.1756	0.3663	0.6011	0.9014	1.6413
Our result σ	0.1756	0.3673	0.5998	0.9292	1.5662
Discrepancy (%)	0.0	−0.3	0.2	−3.1	4.6

The same formulae in terms of the rhombus semiaxes a and b and the aspect ratio $\varepsilon = b/a$ has the form

$$\nu_x = 2\sqrt{2\varepsilon}(1+\varepsilon^2)\left[9\left(1-\varepsilon+\frac{\varepsilon^2}{(1+\varepsilon^2)^{1/2}}\ln\frac{1+\varepsilon+(1+\varepsilon^2)^{1/2}}{1+\varepsilon-(1+\varepsilon^2)^{1/2}}\right)\right]^{-1}$$
$$\nu_y = 2\sqrt{2}(1+\varepsilon^2)\left[9\varepsilon^{3/2}\left(\varepsilon-1+\frac{1}{(1+\varepsilon^2)^{1/2}}\ln\frac{1+\varepsilon+(1+\varepsilon^2)^{1/2}}{1+\varepsilon-(1+\varepsilon^2)^{1/2}}\right)\right]^{-1}. \tag{36}$$

The coefficients of magnetic polarisability of a diamond were computed by de Smedt (1979). We present his results in table 4 compared to those given by formula (36).

The deterioriation of the accuracy of (36) for small values of ε is the result of the erroneous assumption of a square root singularity in (6) which is grossly incorrect for domains with sharp angles.

Table 4.

ε	0.1000	0.2000	0.3333	0.5000	0.7500	0.8000	1.0000
de Smedt ν_x	0.1181	0.1729	0.2341	0.3052	0.4101	0.4323	0.5193
Formula (36) ν_x	0.1078	0.1627	0.2258	0.2986	0.4026	0.4230	0.5043
de Smedt ν_y	6.1820	2.7060	1.5240	0.9946	0.6703	0.6323	0.5193
Formula (36) ν_y	4.5987	2.1982	1.3254	0.9095	0.6388	0.6052	0.5043
Discrepancy of ν_x (%)	8.7	5.9	3.6	2.2	1.8	2.1	2.9
Discrepancy of ν_y (%)	25.6	18.8	13.0	8.6	4.7	4.3	2.9

3.5. *Circular segment*

Let the radius r and the angle 2α be the segment parameters. The location of its centre of gravity is defined by $x_c = kr$, where

$$k = \frac{2\sin^3\alpha}{3(\alpha - \frac{1}{2}\sin 2\alpha)}. \tag{37}$$

The equation of the segment boundary with respect to its centre of gravity takes the form

$$a(\phi) = \begin{cases} r[-k\cos\phi + (1-k^2\sin^2\phi)^{1/2}] & \text{for } 0 \leqslant \phi \leqslant \pi-\gamma \text{ or } \pi+\gamma \leqslant \phi < 2\pi \\ r\dfrac{k-\cos\alpha}{\cos(\pi-\phi)} & \text{for } \pi-\gamma \leqslant \phi \leqslant \pi+\gamma. \end{cases} \tag{38}$$

Computation of the moments yields

$$A = r^2(\alpha - \tfrac{1}{2}\sin 2\alpha) \qquad I_x = \tfrac{1}{4}Ar^2(1-k\cos\alpha) \qquad I_y = \tfrac{1}{4}Ar^2(1+3k\cos\alpha - 4k^2)$$

$$J_x = \tfrac{2}{3}r\left\{-k\sin^3\gamma + (1-k^2\sin^2\gamma)^{1/2}\sin\gamma\cos\gamma + \frac{1-k^2}{k^2}F(\pi-\gamma, k)\right.$$
$$\left. + \frac{2k^2-1}{k^2}E(\pi-\gamma,k) + 3(k-\cos\alpha)\left[-\sin\gamma + \ln\tan\left(\frac{\pi}{4}+\frac{\gamma}{2}\right)\right]\right\}$$

$$J_y=\frac{2}{3}r\left(\sin\gamma[k\sin^2\gamma-3\cos\alpha-(1-k^2\sin^2\gamma)^{1/2}\cos\gamma]\right.$$
$$\left.-\frac{1-k^2}{k^2}F(\pi-\gamma,k)+\frac{1+k^2}{k^2}E(\pi-\gamma,k)\right)$$

where $\gamma=\tan^{-1}(\sin\alpha/(k-\cos\alpha))$. Substituting in (23) leads to

$$\nu_x=\frac{4(1-k\cos\alpha)}{(\alpha-\frac{1}{2}\sin 2\alpha)^{1/2}}\left\{-k\sin^3\gamma+(1-k^2\sin^2\gamma)^{1/2}\sin\gamma\cos\gamma+\frac{1-k^2}{k^2}F(\pi-\gamma,k)\right.$$
$$\left.+\frac{2k^2-1}{k^2}E(\pi-\gamma,k)+3(k-\cos\alpha)\left[-\sin\gamma+\ln\tan\left(\frac{\pi}{4}+\frac{\gamma}{2}\right)\right]\right\}^{-1}$$
$$\nu_y=\frac{4(1+3k\cos\alpha-4k^2)}{(\alpha-\frac{1}{2}\sin 2\alpha)^{1/2}}\left(\sin\gamma[k\sin^2\gamma-3\cos\alpha-(1-k^2\sin^2\gamma)^{1/2}\cos\gamma]\right.$$
$$\left.-\frac{1-k^2}{k^2}F(\pi-\gamma,k)+\frac{1+k^2}{k^2}E(\pi-\gamma,k)\right)^{-1}. \tag{39}$$

A plot of ν_x (full curve) and ν_y (broken curve) against the ratio α/π is given in figure 1. We are unaware of any data to verify the accuracy of (39).

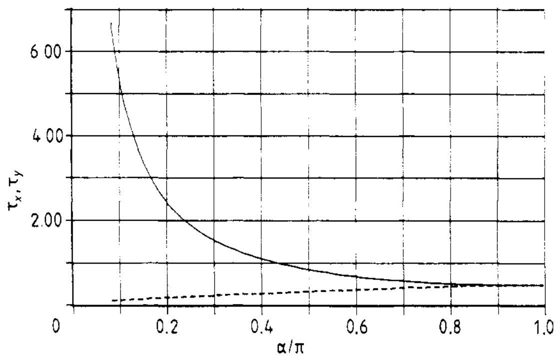

Figure 1. Coefficients of magnetic polarisability for circular segment.

3.6. *Circular sector*

A repetition of the procedure, described in § 3.5, leads to the following results for a circular sector with the angle 2α:

$$A=r^2\alpha \qquad I_x=\tfrac{1}{4}r^4(\alpha-\tfrac{1}{2}\sin 2\alpha) \qquad I_y=r^4\frac{9\alpha^2+9\alpha\sin\alpha\cos\alpha-16\sin^2\alpha}{36\alpha}$$

$$J_x=\frac{2}{3}r\left\{-k\sin^3\gamma-(1-k^2\sin^2\gamma)^{1/2}\sin\gamma\cos\gamma+\frac{1-k^2}{k^2}F(\gamma,k)+\frac{2k^2-1}{k^2}E(\gamma,k)\right.$$
$$\left.+3k\sin\alpha\left[\cos\alpha+\cos(\alpha+\gamma)+\sin^2\alpha\ln\left(\cot\frac{\alpha}{2}\cot\frac{\gamma-\alpha}{2}\right)\right]\right\}$$
$$J_y=\frac{2}{3}r\left\{k\sin\gamma(\sin^2\gamma-3)+(1-k^2\sin^2\gamma)^{1/2}\sin\gamma\cos\gamma\right. \tag{40}$$

$$-\frac{1-k^2}{k^2}F(\gamma,k)+\frac{1+k^2}{k^2}E(\gamma,k))$$
$$+3k\sin\alpha\left[-\cos\alpha-\cos(\alpha+\gamma)+\cos^2\alpha\ln\left(\cot\frac{\alpha}{2}\cot\frac{\gamma-\alpha}{2}\right)\right]\Bigg\}.$$

Here, $k=2\sin\alpha/(3\alpha)$ and $\gamma=\tan^{-1}[\sin\alpha/(\cos\alpha-k)]$. The coefficients sought are expressed as follows:

$$\nu_x=2\alpha^{-3/2}(2\alpha-\sin 2\alpha)\Bigg\{-k\sin^3\gamma-(1-k^2\sin^2\gamma)^{1/2}\sin\gamma\cos\gamma$$
$$+\frac{1-k^2}{k^2}F(\gamma,k)+\frac{2k^2-1}{k^2}E(\gamma,k)$$
$$+3k\sin\alpha\left[\cos\alpha+\cos(\alpha+\gamma)+\sin^2\alpha\ln\left(\cot\frac{\alpha}{2}\cot\frac{\gamma-\alpha}{2}\right)\right]\Bigg\}^{-1} \tag{41}$$
$$\nu_y=\frac{4(9\alpha^2+9\alpha\sin\alpha\cos\alpha-16\sin^2\alpha)}{9\alpha^{5/2}}\Bigg\{k\sin\gamma(\sin^2\gamma-3)$$
$$+(1-k^2\sin^2\gamma)^{1/2}\sin\gamma\cos\gamma-\frac{1-k^2}{k^2}F(\gamma,k)+\frac{1+k^2}{k^2}E(\gamma,k)$$
$$+3k\sin\alpha\left[-\cos\alpha-\cos(\alpha+\gamma)+\cos^2\alpha\ln\left(\cot\frac{\alpha}{2}\cot\frac{\gamma-\alpha}{2}\right)\right]\Bigg\}^{-1}.$$

Formulae (41) are exact for a complete circle ($\alpha=\pi$), and give the same results as (39) for a half-circle ($\alpha=\pi/2$). The plot of ν_x (full curve) and ν_y (broken curve) against the ratio α/π is given in figure 2. We did not find anything in the literature to compare with these results.

3.7. Cross

Consider an aperture obtained by an orthogonal intersection of two equal rectangles with sides $2a$ and $2b$. Introduce the aspect ratio as $\varepsilon=b/a$. The area and the moments will take the form

$$A=4a^2\varepsilon(2-\varepsilon) \qquad I_x=I_y=\tfrac{4}{3}a^4\varepsilon(1+\varepsilon^2-\varepsilon^3)$$
$$J_x=J_y=4a\left(\ln[\varepsilon+(1+\varepsilon^2)^{1/2}]+\varepsilon\ln\frac{1+(1+\varepsilon^2)^{1/2}}{(1+\sqrt{2})\varepsilon}\right).$$

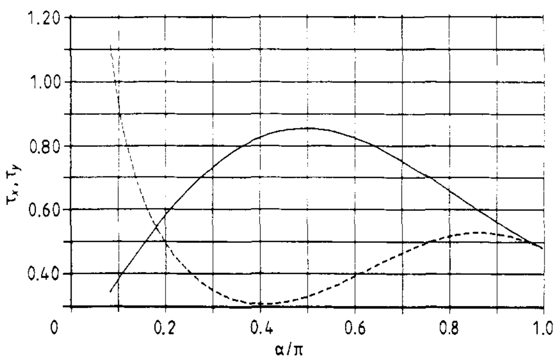

Figure 2. Coefficients of magnetic polarisability for circular sector.

The coefficients will be defined as

$$\nu_x = \nu_y = \frac{4\varepsilon(1+\varepsilon^2-\varepsilon^3)}{9\varepsilon(2-\varepsilon)^{3/2}}\left(\ln[\varepsilon+(1+\varepsilon^2)^{1/2}]+\varepsilon\ln\frac{1+(1+\varepsilon^2)^{1/2}}{(1+\sqrt{2})\varepsilon}\right)^{-1}. \tag{42}$$

The comparison between the results due to (42) and those given by de Smedt (1979) are presented in table 5. Taking into consideration the shape complexity, we should consider the agreement of these results as surprisingly good, not only quantitatively but qualitatively as well: both data display a relatively flat minimum around $\varepsilon = 0.75$.

Table 5.

ε	0.1000	0.2000	0.3333	0.4000	0.5000	0.6000	0.7500	0.8000	1.0000
de Smedt $\nu_x = \nu_y$	1.5910	0.8720	0.6255	0.5725	0.5267	0.5069	0.4985	0.4997	0.5193
Formula (42) $\nu_x = \nu_y$	1.7382	0.8758	0.6006	0.5465	0.5049	0.4890	0.4893	0.4926	0.5043
Discrepancy (%)	−9.3	−0.4	4.0	4.5	4.1	3.5	1.9	1.4	2.9

4. Discussion

It is noteworthy that the change of the order of integration which led to (4) is valid inside the circle $\rho \leqslant \min\{a(\phi)\}$ only, and this explains the accuracy deterioration for the aspect ratio far away from unity. Nevertheless, one can obtain from (4) the *exact* solution for an ellipse and sufficiently accurate formulae for various specific apertures as was demonstrated in the previous section.

The accuracy of formulae (23) can be improved by taking into consideration the fifth harmonic (12) in combination with the variational approach (Noble 1960). The following functional assumes its maximum value at the exact solution of (1):

$$I(\sigma) = 2\iint_S \sigma(M)w(M)\,\mathrm{d}S_M - \iint_S \sigma(M)\left(\iint_S \frac{\sigma(N)}{R(M,N)}\,\mathrm{d}S_N\right)\mathrm{d}S_M. \tag{43}$$

Taking

$$\iint_S \frac{\sigma(N)}{R(M,N)}\,\mathrm{d}S_N \approx w_1 + w_5 \tag{44}$$

and substituting (6), (10), (12) and (44) in (43), one gets after integration with respect to ρ

$$\begin{aligned} I = \int_0^{2\pi} & (a(\phi))^4\{(p_1\cos\phi + p_2\sin\phi)[\tfrac{4}{3}(\alpha_x\sin\phi - \alpha_y\cos\phi) \\ & -\tfrac{1}{3}\pi(p_1J_y + p_2J_{xy})\cos\phi - \tfrac{1}{3}\pi(p_1J_{xy}+p_2J_x)\sin\phi \\ & -\tfrac{4}{63}(a(\phi))^3([p_1(A_{c6}+A_{c4})+p_2(A_{s6}-A_{s4})]\cos 5\phi \\ & +[p_1(A_{s6}+A_{s4})+p_2(A_{c4}-A_{c6})]\sin 5\phi)]\}\,\mathrm{d}\phi. \end{aligned} \tag{45}$$

Considering now the functional I as a function of p_1 and p_2, the extremum conditions

$$\partial I/\partial p_1 = 0 \qquad \partial I/\partial p_2 = 0$$

give two linear algebraic equations with respect to the unknowns p_1 and p_2. The complete solution is pretty cumbersome. Here, we present the final result for the coefficients ν_x and ν_y which are valid only for domains having at least one axis of symmetry and the central principal axes taken as the coordinate axes:

$$\nu_x = \frac{32 I_x}{3A^{3/2} J_x (1+\eta_x)} \qquad \nu_y = \frac{32 I_y}{3A^{3/2} J_y (1+\eta_y)} \tag{46}$$

where the correction terms

$$\eta_x = \frac{(B_{c4} - B_{c6})(A_{c4} - A_{c6})}{42\pi I_x J_x} \qquad \eta_y = \frac{(B_{c4} + B_{c6})(A_{c4} + A_{c6})}{42\pi I_y J_y} \tag{47}$$

and

$$B_{c6} = \int_0^{2\pi} (a(\phi))^7 \cos 6\phi \, \mathrm{d}\phi \qquad B_{c4} = \int_0^{2\pi} (a(\phi))^7 \cos 4\phi \, \mathrm{d}\phi.$$

Since expression (44) is approximate, there is no guarantee that (46) will be more accurate than (23). We performed the necessary computations for a rectangle. In table 6 the results are compared to those by de Smedt (1979).

Comparison with similar data computed on the basis of formula (32) shows that the correction terms η_x and η_y in this particular case resulted in decreasing of the value of discrepancy, positive as well as negative. We caution again that there is no guarantee that this will be valid for an arbitrary domain. For example, in table 7 the data are computed for a rhombus.

Table 6.

ε	0.1000	0.2000	0.3333	0.5000	0.7500	0.8000	1.0000
de Smedt ν_x	0.1287	0.1881	0.2531	0.3249	0.4240	0.4436	0.5193
Formula (46) ν_x	0.1405	0.1988	0.2577	0.3207	0.4165	0.4376	0.5331
de Smedt ν_y	4.1070	2.0260	1.2600	0.8892	0.6426	0.6130	0.5193
Formula (46) ν_y	4.5856	2.0985	1.2479	0.8714	0.6463	0.6190	0.5331
Discrepancy in ν_x (%)	−9.2	−5.7	−1.8	1.3	1.8	1.3	−2.7
Discrepancy in ν_y (%)	−11.7	−3.6	1.0	2.0	−0.6	−1.0	−2.7

Table 7.

ε	0.1000	0.2000	0.3333	0.5000	0.7500	0.8000	1.0000
de Smedt ν_x	0.1181	0.1729	0.2341	0.3052	0.4101	0.4323	0.5193
Formula (46) ν_x	0.2268	0.1860	0.2351	0.3031	0.4058	0.4264	0.5091
de Smedt ν_y	6.1820	2.7060	1.5240	0.9946	0.6703	0.6323	0.5193
Formula (46) ν_y	8.5600	2.5916	1.4196	0.9408	0.6490	0.6138	0.5091
Discrepancy of ν_x (%)	−92.0	−7.6	−0.4	0.7	1.0	1.4	2.0
Discrepancy of ν_y (%)	−38.5	4.2	6.8	5.4	3.2	2.9	2.0

Comparison with the data computed due to (36) indicates that the discrepancy decreased for $\varepsilon \geqslant 0.2$ while for $\varepsilon = 0.1$ it has jumped in the opposite direction to −92%. The main reason for this is a jump in the value of the coefficients η_x and η_y when ε is very small. The following rule of thumb may be suggested for the user wishing to improve the accuracy: when the value of the correction coefficients η_x and η_y does not exceed a small percentage of unity this generally means an improvement in accuracy, otherwise one should not use formulae (46).

It is worthwhile giving the solution due to (45) for the case when the aperture has no axis of symmetry and only the first harmonic of w_1 is taken into consideration. The result is

$$p_1 = \frac{\alpha_x(c_{22}I_{xy} - c_{12}I_x) + \alpha_y(c_{12}I_{xy} - c_{22}I_y)}{c_{11}c_{22} - c_{12}^2}$$
$$p_2 = \frac{\alpha_x(c_{11}I_x - c_{12}I_{xy}) + \alpha_y(c_{12}I_y - c_{11}I_{xy})}{c_{11}c_{22} - c_{12}^2} \tag{48}$$

where

$$c_{11} = \tfrac{1}{2}\pi(J_yI_y + J_{xy}I_{xy}) \qquad c_{22} = \tfrac{1}{2}\pi(J_xI_x + J_{xy}I_{xy})$$
$$c_{12} = \tfrac{1}{4}\pi[J_{xy}(I_x + I_y) + I_{xy}(J_x + J_y)].$$

Formulae (48) look different from the equivalent set (15) derived earlier. In the absence of any numerical data related to a general domain, it is impossible to say whether formulae (48) are more accurate than (15), but they are definitely more complicated. It is noteworthy that in the case of a domain with an axis of symmetry both (48) and (15) simplify to the same equations (18).

5. Conclusion

Formulae (22) and (23) proved simple and effective for evaluating the coefficients of magnetic polarisability of small apertures having at least one axis of symmetry. Their high accuracy is confirmed by numerous examples. Though the accuracy deteriorates for domains with sharp angles and the aspect ratio far away from unity, we believe that the method presented in this paper will provide a useful tool easily accessible to a practical engineer. The computation of the coefficient of electrical polarisability will be considered in the second part of this project. Results of this paper are useful for the solution of mathematically similar problems in the other branches of engineering science (Fluid and Solid Mechanics, Acoustics, Heat Transfer, etc).

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Appendix

Here we repeat the derivation leading to the integral representation for the reciprocal distance (2), as was given in Fabrikant (1971). Consider the expression

$$\frac{1}{R^{1+u}} = \frac{1}{[\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi - \phi_0)]^{(1+u)/2}} \tag{A1}$$

where u is a constant and $-1 < u < 1$. The standard expansion of (A1) in Fourier series will take the form

$$\frac{1}{[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)]^{(1+u)/2}}$$

$$= \sum_{n=-\infty}^{\infty} \frac{\exp[in(\phi-\phi_0)]}{2\pi} \int_0^{2\pi} \frac{\exp(-in\psi)\,d\psi}{(\rho^2+\rho_0^2-2\rho\rho_0\cos\psi)^{(1+u)/2}}$$

$$= \sum_{n=-\infty}^{\infty} \frac{\exp[in(\phi-\phi_0)]}{2\pi\rho_0^{1+u}} \frac{2\pi\Gamma[n+(1+u)/2]}{\Gamma[(1+u)/2]\Gamma(n+1)}\left(\frac{\rho}{\rho_0}\right)^n$$

$$\times F\left(\frac{1+u}{2}, n+\frac{1+u}{2}, n+1; \frac{\rho^2}{\rho_0^2}\right). \tag{A2}$$

Here F stands for the Gauss hypergeometric function. By using another integral representation

$$F\left(\frac{1+u}{2}, n+\frac{1+u}{2}, n+1; z\right)$$

$$= \frac{2\Gamma(n+1)}{\Gamma[n+(1+u)/2]\Gamma[1-(1+u)/2]} \int_0^1 \frac{t^{2n+u}(1-t^2)^{-(1+u)/2}}{(1-zt^2)^{(1+u)/2}}\,dt$$

expression (A2) can be transformed into

$$\frac{2}{\pi}\cos\frac{\pi u}{2} \sum_{n=-\infty}^{\infty} \frac{\exp[in(\phi-\phi_0)]}{(\rho\rho_0)^n} \int_0^{\min(\rho_0,\rho)} \frac{x^{2n+u}\,dx}{[(\rho^2-x^2)(\rho_0^2-x^2)]^{(1+u)/2}}. \tag{A3}$$

Summation in (A3) finally gives

$$\frac{1}{[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)]^{(1+u)/2}}$$

$$= \frac{2}{\pi}\cos\frac{\pi u}{2} \int_0^{\min(\rho_0,\rho)} \frac{\lambda(x^2/\rho\rho_0, \phi-\phi_0)x^u\,dx}{[(\rho^2-x^2)(\rho_0^2-x^2)]^{(1+u)/2}}. \tag{A4}$$

In the particular case $u=0$ formula (A4) gives the required representation (2).

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